Fredholm Operators

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Abstract.

The Fredholm index is an indispensable item in the operator theorist's toolkit and a very simple prototype of the application of algebraic topological methods to analysis. This document contains all the basics one needs to know to work with Fredholm theory, as well as many examples. The actual talk that I gave only dealt with the essentials and I omitted most of the proofs presented here. At the end, I present my solutions to some of the exercises in *Murphy's C^*-algebras and operator theory* that deal with compact operators and Fredholm theory.

A (not so brief) review of compact operators.

Let X be a topological space. Recall that a subset $Y \subseteq X$ is said to be **relatively compact** if $\overline{Y}$ is compact in X. Recall also that a subset $Y \subseteq X$ is said to be **totally bounded** if $\forall\ \varepsilon > 0,\ \exists\ n \in \mathbb{N}$, and $x_1, \dots, x_n \in X$ such that

$$Y \subseteq \bigcup_{k=1}^{n} B_\varepsilon(x_k)$$

Definition. A linear map $u : X \to Y$ between Banach spaces X and Y is said to be **compact** if $u(S)$ is relatively compact in Y, where $S := \{x \in X : \|x\| \leq 1\} = \overline{B_1(0)}$. ▲

Proposition. *Let (X, d) be a complete metric space. Then $Y \subseteq X$ is relatively compact if and only if Y is totally bounded.*

Proof. Assume first that Y is relatively compact. Let $\varepsilon > 0$. Then, $\{B_\varepsilon(x)\}_{x\in\overline{Y}}$ is an open cover for $\overline{Y}$. Since $\overline{Y}$ is compact, it follows that there are $n \in \mathbb{N}$, $x_1, \ldots, x_n \in \overline{Y}$ such that $\{B_\varepsilon(x_k)\}_{k=1}^n$ is an open cover for $\overline{Y}$. Hence,

$$Y \subseteq \overline{Y} \subseteq \bigcup_{k=1}^n B_\varepsilon(x_k),$$

and therefore Y is totally bounded.

Conversely, we now suppose that Y is totally bounded. We claim that $\overline{Y}$ is also totally bounded. Indeed, let $\varepsilon > 0$, since Y is totally bounded there are $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$ such that

$$Y \subseteq \bigcup_{k=1}^n B_{\varepsilon/2}(x_k),$$

whence

$$\overline{Y} \subseteq \bigcup_{k=1}^n \overline{B_{\varepsilon/2}(x_k)} \subset \bigcup_{k=1}^n B_\varepsilon(x_k),$$

proving the claim. Next, we show that $\overline{Y}$ is compact by proving that any sequence $(y_k)_{k=1}^\infty$ in $\overline{Y}$ admits a convergence subsequence. Since X is a complete metric space and $\overline{Y}$ is closed, it suffices to find a Cauchy subsequence. Using that $\overline{Y}$ is totally bounded, we find for each $n \in \mathbb{N}$ a finite set $F_n \subset X$ such that

$$\overline{Y} \subseteq \bigcup_{x\in F_n} B_{1/n}(x),$$

Since $(y_k)_{k=1}^\infty$ is a sequence in $\overline{Y} \subseteq \bigcup_{x\in F_1} B_1(x)$, there must be $x_1 \in F_1$ such that infinitely many terms of $(y_k)_{k=1}^\infty$ are in $B_1(x_1)$. Let

$$J_1 := \{k \in \mathbb{N} : y_k \in B_1(x_1)\}.$$

By definition, $(y_k)_{k\in J_1}$ is a subsequence of $(y_k)_{k=1}^\infty$. Similarly, there must be $x_2 \in F_2$ such that such that infinitely many terms of $(y_k)_{k\in J_1}$ are in $B_{1/2}(x_2)$. Let

$$J_2 := \{k \in J_1 : y_k \in B_{1/2}(x_2)\}.$$

We keep doing thus and get, at the j-th step an element $x_j \in F_j$ and an infinite set

$$J_j := \{k \in J_{j-1} : y_k \in B_{1/j}(x_j)\}.$$

Further, $J_1 \supseteq J_2 \supseteq \ldots \supseteq J_j \supseteq \ldots$ is a decreasing sequence of infinite subsets of $\mathbb{N}$. Thus, for each $j \in \mathbb{N}$ we can pick $k_j \in J_j$ such that $k_1 < k_2 < \ldots <$

$k_j < \ldots$. This gives a sequence $(y_{k_j})_{j=1}^{\infty}$ which is clearly a subsequence of $(y_k)_{k=1}^{\infty}$. Finally, since $y_{k/j} \in B_{1/j}(x_j)$ it follows that for any $j < l$

$$d(y_{k_j}, y_{k_l}) \leq d(y_{k_j}, x_j) + d(x_j, y_{k_l}) < \frac{1}{j} + \frac{1}{j} = \frac{2}{j}.$$

Thus, $(y_{k_j})_{j=1}^{\infty}$ is Cauchy, as we wanted to show. ■

Corollary. *A linear map $u : X \to Y$ between Banach spaces X and Y is compact if and only if $u(S)$ is totally bounded.*

Remark. If $u : X \to Y$ is compact, then $u(S)$ is totally bounded and therefore bounded. Thus,

$$\|u\| = \sup_{x \in S} \|u(x)\| < \infty$$

Hence, $u \in \mathcal{B}(X, Y)$. We will denote by $\mathcal{K}(X, Y)$ to the set of compact operators from X to Y. When $X = Y$ we just write $\mathcal{K}(X)$. We will show that $\mathcal{K}(X)$ is a closed ideal of $\mathcal{B}(X)$. ▼

Example. • For $k \in C([0,1]^2)$, we define $u : C([0,1]) \to C([0,1])$ by

$$u(f)(s) := \int_0^1 k(t,s) f(t) dt \quad \forall \ f \in C([0,1]), s \in [0,1]$$

Then, one can use Arzela-Ascoli's theorem to show that $u \in \mathcal{K}(C([0,1]))$.

• Similarly, if we define $v : C([0,1]) \to C([0,1])$ by

$$v(f)(s) := \int_0^s f(t) dt \quad \forall \ f \in C([0,1]), s \in [0,1],$$

we also get $v \in \mathcal{K}(C([0,1]))$. This is not a particular case of the first example, since the kernel for v is given by $k(t,s) := \chi_{[0,s]}(t)$, which is not a continuous map.

• In general, take $(X, \mathfrak{M}, \mu)$ a measure space, $1 < p < \infty$, q the Hölder conjugate for p and $k : X \times X \to \mathbb{C}$ a measurable function on $\mathfrak{M} \otimes \mathfrak{M}$ such that

$$\int_X \left(\int_X |k(x,y)|^p d\mu(x) \right)^{q/p} d\mu(y) < \infty.$$

Then, if we define $w : L^p(X, \nu) \to L^p(X, \mu)$

$$w(f)(x) = \int_X k(x,y) f(y) d\mu(y) \quad \forall \ L^p(X, \mu), x \in X,$$

we can show that $w \in \mathcal{K}(L^p(X, \mu))$. ▼

The following theorem gives a useful alternative characterizations for compact operators.

Theorem. *Let X, Y be Banach spaces and $u \in \mathcal{B}(X,Y)$. The following are equivalent:*

(i) $u \in \mathcal{K}(X,Y)$.

(ii) *For each bounded set $B \subseteq X$, $u(B)$ is relatively compact in Y.*

(iii) *If $(x_n)_{n=1}^{\infty}$ is a bounded sequence in X, then $(u(x_n))_{n=1}^{\infty}$ admits a convergent subsequence in Y.*

Proof.

(i) $\Rightarrow$ *(ii)* Suppose $u : X \to Y$ is compact. Let $B \subset X$ be a bounded set. Then, there is an $M > 0$ such that $B \subset \overline{B_M(0)}$. Notice that $\overline{B_M(0)} = M\overline{B_1(0)}$, so since u is a linear map it follows that $u(B) \subset Mu(S)$, where as before $S = \overline{B_1(0)}$. Since u is compact, $u(S)$ is relatively compact and therefore $\overline{u(B)}$ is a closed subset of a compact set, whence compact. That is, $u(B)$ is relatively compact in Y.

(ii) $\Rightarrow$ *(iii)* We now assume that for each bounded set $B \subseteq X$, $u(B)$ is relatively compact in Y. Let $(x_n)_{n=1}^{\infty}$ is a bounded sequence in X. Then, $B := \{x_n : n \in \mathbb{N}\}$ is a bounded subset of X. Thus, $\overline{u(B)}$ is compact in Y. Since $(u(x_n))_{n=1}^{\infty}$ is a sequence in $\overline{u(B)}$, it follows that it admits a convergent subsequence in Y.

(iii) $\Rightarrow$ *(i)* Finally, we assume that any bounded sequence in X is sent by u into a sequence that admits a convergent subsequence in Y. We show that $\overline{u(S)}$ is compact in Y. Let $(y_n)_{n=1}^{\infty}$ be any sequence in $\overline{u(S)}$. For each $n \in \mathbb{N}$, we find $x_n \in S$ such that

$$\|y_n - u(x_n)\| < \frac{1}{n}$$

Then, $(x_n)_{n=1}^{\infty}$ is a sequence in S and therefore bounded. Thus, $(u(x_n))_{n=1}^{\infty}$ admits a convergent subsequence, say $(u(x_{n_k}))_{k=1}^{\infty}$ such that $u(x_{n_k}) \to y \in Y$ as $k \to \infty$. Then,

$$\|y_{n_k} - y\| \leq \|y_{n_k} - u(x_{n_k})\| + \|u(x_{n_k}) - y\| \underset{k\to\infty}{\longrightarrow} 0$$

Thus, $(y_n)_{n=1}^{\infty}$ admits a convergent subsequence and therefore $\overline{u(S)}$ is compact.

■

Corollary. *If $u, v \in \mathcal{K}(X,Y)$ and $\lambda \in \mathbb{C}$, then $u + \lambda v \in \mathcal{K}(X,Y)$. That is, $\mathcal{K}(X,Y)$ is a vector subspace of $\mathcal{B}(X,Y)$.*

Proof. If $(x_n)_{n=1}^{\infty}$ is any bounded sequence in X, then compactness of u implies that there is a subsequence $(x_{n_k})_{k=1}^{\infty}$ such that $(u(x_{n_k}))_{k=1}^{\infty}$ converges. Since $(x_{n_k})_{k=1}^{\infty}$ is still a bounded sequence, compactness of v implies now that there is a subsequence $(x_{n_l})_{l=1}^{\infty}$ such that both $(u(x_{n_l}))_{l=1}^{\infty}$ and $(v(x_{n_l}))_{l=1}^{\infty}$ converge. Hence, $((u+\lambda v)(x_{n_l}))_{l=1}^{\infty}$ is converges and therefore $u + \lambda v$ is a compact operator. ■

Corollary. *Let X' and Y' be also Banach spaces. If $u \in \mathcal{K}(X,Y)$, $v \in \mathcal{B}(X',X)$, and $w \in \mathcal{B}(Y,Y')$, then $uv \in \mathcal{K}(X',Y)$ and $wu \in \mathcal{K}(X,Y')$. In particular this gives that $\mathcal{K}(X)$ is an ideal of $\mathcal{B}(X)$.*

Proof. First, let $(x'_n)_{n=1}^{\infty}$ be any bounded sequence, say by $M > 0$, in X'. Then, $(v(x'_n))_{n=1}^{\infty}$ is a bounded sequence, by $M\|v\|$, in X and therefore, by compactness of u, $(uv(x'_n))_{n=1}^{\infty}$ admits a convergent subsequence. Thus, uv is compact.

Second, if $(x_n)_{n=1}^{\infty}$ is any bounded sequence in X, by compactness of u we have that $(u(x_n))_{n=1}^{\infty}$ admits a convergent subsequence, say $(u(x_{n_k}))_{k=1}^{\infty}$, converging to an element $y \in Y$. Since

$$\|wu(x_{n_k}) - w(y)\| \leq \|w\|\|u(x_{n_k}) - y\|,$$

it follows that $(wu(x_{n_k}))_{k=1}^{\infty}$ converges to $w(y)$ and therefore wu is compact. ■

So far we know that $\mathcal{K}(X)$ is an ideal of $\mathcal{B}(X)$, but one could ask whether it is a proper ideal or a closed one. The following two theorems give answers to this.

Theorem. $\mathcal{K}(X) = \mathcal{B}(X)$ *if and only if X is finite dimensional*

Proof. Recall that X is finite dimensional if and only if $S := \overline{B_1(0)}$ is compact. Clearly, S is compact if and only if id_X is a compact operator. Thus, it's enough to show that $\mathcal{K}(X) = \mathcal{B}(X)$ if and only if id_X is a compact operator. Assume first that $\mathcal{K}(X) = \mathcal{B}(X)$. Then, obviously $\mathrm{id}_X \in \mathcal{K}(X)$. Conversely, suppose that $\mathrm{id}_X \in \mathcal{K}(X)$. Then, since $\mathcal{K}(X)$ is an ideal of $\mathcal{B}(X)$, it follows that any $a \in \mathcal{B}(X)$ is also in $\mathcal{K}(X)$ because $a = \mathrm{id}_X a$. Hence, $\mathcal{K}(X) = \mathcal{B}(X)$. ■

Theorem. *$\mathcal{K}(X,Y)$ is a closed subset of $\mathcal{B}(X,Y)$.*

Proof. Let $(u_n)_{n=1}^\infty$ be a sequence in $\mathcal{K}(X,Y)$ that converges to some $a \in \mathcal{B}(X,Y)$. We have to show that $a \in \mathcal{K}(X,Y)$. Let $(x_n)_{n=1}^\infty$ be any bounded sequence, say by $M > 0$, in X. Since u_1 is compact, $(x_n)_{n=1}^\infty$ admits a subsequence $(x_{k_1})_{k_1=1}^\infty$ such that $(u_1(x_{k_1}))_{k_1=1}^\infty$ is convergent. Since $(x_{k_1})_{k_1=1}^\infty$ is also bounded by M and u_2 is compact, $(x_{k_1})_{k_1=1}^\infty$ admits a subsequence $(x_{k_2})_{k_2=1}^\infty$ such that both $(u_1(x_{k_2}))_{k_2=1}^\infty$ and $(u_2(x_{k_2}))_{k_2=1}^\infty$ are convergent. Continuing this way, we get "nested subsequences"

$$(x_n)_{n=1}^\infty \supset (x_{k_1})_{k_1=1}^\infty \supset (x_{k_2})_{k_2=1}^\infty \supset \ldots \supset (x_{k_j})_{k_j=1}^\infty \supset \ldots$$

such that $(\ u_l(x_{k_j})\ )_{k_j=1}$ is convergent for any $l \leq j$. We define a sequence $(z_m)_{m=1}^\infty$ by letting z_m be the m-th term of the sequence $(x_{k_m})_{k_m=1}^\infty$. Hence, $(z_m)_{m=1}^\infty$ is a subsequence of $(x_n)_{n=1}^\infty$. We claim that $(a(z_m))_{m=1}^\infty$ is a convergent sequence in Y, and therefore it will follow that $a \in \mathcal{K}(X,Y)$. To prove the claim, it suffices to show that $(a(z_m))_{m=1}^\infty$ is Cauchy. Let $\varepsilon > 0$. Since $u_n \to a$, there is an $N \in \mathbb{N}$ such that $\|u_n - a\| < \frac{\varepsilon}{3M}$ for all $n \geq N$. Notice that $(z_m)_{m=N}^\infty$ is a subsequence of $(x_{K_N})_{K_N=1}^\infty$ and therefore $(u_N(z_m))_{m=N}^\infty$ is convergent. Therefore, there is an $N' \in \mathbb{N}_{\geq N}$ such that

$$\|u_N(z_m) - u_N(z_{m'})\| < \frac{\varepsilon}{3}$$

for all $m, m' \geq N'$. Thus, if $m, m' \geq N'$.

$$\begin{aligned}\|a(z_m) - a(z_{m'})\| &\leq \|a(z_m) - u_N(z_m)\| + \|u_N(z_m) - u_N(z_{m'})\| + \|u_N(z_{m'}) - a(z_m')\| \\ &\leq \|a - u_N\|\|z_m\| + \|u_N(z_m) - u_N(z_{m'})\| + \|u_N - a\|\|z_{m'}\| \\ &< \frac{\varepsilon}{3M}M + \frac{\varepsilon}{3} + \frac{\varepsilon}{3M}M = \varepsilon\end{aligned}$$

So $(a(z_m))_{m=1}^\infty$ is indeed Cauchy. ■

Remark. • Notice that any bounded linear map with finite rank is compact. Indeed, since in finite dimensional vector spaces compactness is equivalent to closed and bounded, if $u(X)$ is finite dimensional and $B \subset X$ is bounded, it follows that $\overline{u(B)}$ is closed and bounded in $u(X)$ and therefore compact.

• Thus, combining the previous remark and our last theorem, we conclude that the norm limit of finite rank operators is always a compact operator. In Hilbert spaces, the converse is true: Any element of $\mathcal{K}(\mathcal{H})$ is norm limit of finite rank operators. However, for arbitrary Banach spaces the converse may not hold. It's only true for those Banach spaces that have a Shauder basis (i.e. there is (b_n) in X such that for any $x \in X$ there is (λ_n) in $\mathbb{C}$ such that $x = \sum_n \lambda_n b_n$ in norm). ▼

Recall that the dual space of a Banach space X, defined as $X^* := \mathcal{B}(X, \mathbb{C})$, is the set of bounded linear functionals on X. For any $a \in \mathcal{B}(X, Y)$ we get $a^* : Y^* \to X^*$ by letting $a^*(\varphi) := \varphi \circ a \;\; \forall\, \varphi \in Y^*$. Then, since

$$|a^*(\varphi)(x)| = |\varphi(a(x))| \leq \|\varphi\| \|a\| \|x\|,$$

it follows that $\|a^*\| \leq \|a\|$ and therefore $a^* \in \mathcal{B}(Y^*, X^*)$.

Theorem. *If $u \in \mathcal{K}(X, Y)$, then $u^* \in \mathcal{K}(Y^*, X^*)$*

Proof. Let as usual $S := \overline{B_1(0)} \subset X$ and let $T := \overline{B_1(0)} \subset Y^*$. We only need to show that $u^*(T)$ is totally bounded. Let $\varepsilon > 0$. Since $u(S)$ is totally bounded, there is $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in S$ such that

$$u(S) \subseteq \bigcup_{k=1}^{n} B_{\varepsilon/3}(u(x_k))$$

Hence, for any $u(x) \in u(S)$, there is $1 \leq i \leq n$ such that $\|u(x) - u(x_i)\| < \frac{\varepsilon}{3}$. We use this to define $v : Y^* \to \mathbb{C}^n$ by

$$v(\varphi) := (\; \varphi(u(x_1)), \varphi(u(x_2)), \ldots, \varphi(u(x_n)) \;)$$

One checks that v is a bounded linear map, and furthermore $v \in \mathcal{K}(Y^*, \mathbb{C}^n)$, because v has finite rank. Thus, $v(T)$ is totally bounded. That is, there is $m \in \mathbb{N}$ and $\varphi_1, \ldots, \varphi_m \in T$ such that for any $v(\varphi) \in v(T)$, there is $1 \leq j \leq m$ so that $\|v(\varphi) - v(\varphi_j)\| < \frac{\varepsilon}{3}$. Notice that

$$\begin{aligned}
\|v(\varphi) - v(\varphi_j)\| &= \max_{1 \leq k \leq n} \{|\varphi(u(x_k)) - \varphi_j(u(x_k))|\} \\
&= \max_{1 \leq k \leq n} \{|u^*(\varphi)(x_k) - u^*(\varphi_j)(x_k)|\} \\
&\geq |u^*(\varphi)(x_i) - u^*(\varphi_j)(x_i)|
\end{aligned}$$

Hence, we have $\|u(x) - u(x_i)\| < \frac{\varepsilon}{3}$ and also $|u^*(\varphi)(x_i) - u^*(\varphi_j)(x_i)| < \frac{\varepsilon}{3}$. Therefore, since $\varphi, \varphi_j \in T$ we have

$$\begin{aligned}
&|u^*(\varphi)(x) - u^*(\varphi_j)(x)| \\
&\quad \leq |\varphi(u(x) - u(x_i))| + |u^*(\varphi)(x_i) - u^*(\varphi_j)(x_i)| + |\varphi_j(u(x_i) - u(x))| \\
&\quad \leq \|u(x) - u(x_i)\| + |u^*(\varphi)(x_i) - u^*(\varphi_j)(x_i)| + \|u(x_i) - u(x)\| \\
&\quad < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varphi
\end{aligned}$$

Since $x \in S$, this gives $\|u^*(\varphi) - u^*(\varphi_j)\| < \varepsilon$, so $u^*(T)$ is indeed totally bounded. ■

Fredholm Theory

Preliminaries

Definition. A linear map $u : X \to Y$ between Banach spaces is said to be **bounded below** if there is $\delta > 0$ such that

$$\|u(x)\| \geq \delta \|x\| \quad \forall\, x \in X$$

▲

Lemma. *If $u \in \mathcal{B}(X, Y)$ is bounded below, then $u(X)$ is closed in Y.*

Proof. Let $(u(x_n))_{n=1}^{\infty}$ be a sequence in $u(X)$ that converges to some $y \in Y$. We shall prove that $y \in u(X)$. Since u is bounded below, there is a $\delta > 0$ such that if $m, n \in \mathbb{N}$, then we have

$$\|x_m - x_n\| \leq \frac{1}{\delta}\|u(x_m) - u(x_n)\|$$

In particular, this gives that $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in X and therefore converges to a point $x \in X$. Finally, since

$$\|u(x) - y\| \leq \|u(x) - u(x_n)\| + \|u(x_n) - y\| \leq \|u\|\|x - x_n\| + \|u(x_n) - y\|,$$

it follows at once that $y = u(x) \in u(X)$. ■

Example. • If $u \in \mathcal{B}(X, Y)$ is invertible, then for any $x \in X$ we have

$$\|x\| = \|u^{-1}(u(x))\| \leq \|u^{-1}\|\|u(x)\|$$

Hence, $\|u(x)\| \geq \frac{1}{\|u^{-1}\|}\|x\|$, which gives that u is bounded below.

• If $u \in \mathcal{B}(X, Y)$ is an isometry, then u is clearly bounded below. ▼

Lemma. *A map $u \in \mathcal{B}(X, Y)$ is not bounded below if and only if there is a sequence of unit vectors $(x_n)_{n=1}^{\infty}$ in X such that $\lim_{n\to\infty} u(x_n) = 0$*

Proof. Suppose first that u is not bounded below. Then, there is $x \in X$ such that $\|u(x)\| < \delta\|x\|$ for all $\delta > 0$. In particular this says that $u(x) = 0$. Hence the sequence $x_n := x/\|x\|$ is the required one.

Conversely, assume that u is bounded below. Then, there is $\delta > 0$ such that $\|u(x)\| \geq \delta\|x\|$ for all $x \in X$. In particular, if $(x_n)_{n=1}^{\infty}$ is any sequence of unit vectors we have

$$\|u(x_n)\| \geq \delta\|x_n\| = \delta > 0$$

So it's impossible to have $\lim_{n\to\infty} u(x_n) = 0$. ■

Definition. A subspace Y of a Banach space X is called **a complemented subspace** if there exist a linear bounded operator $p : X \to Y$ such that $p(X) = Y$ and $p(y) = y$ for all $y \in Y$. ▲

Lemma. *A subspace Y of a Banach space X is a complemented subspace if and only if there exists a closed subspace Z of X such that $X = Y \oplus Z$.*

Proof. Suppose first that Y is a complemented subspace of X. Then, there is a linear bounded operator $p : X \to Y$ such that $p(X) = Y$ and $p(y) = y$ for all $y \in Y$. Define $Z := \ker(p)$, it's clear that Z is a closed subspace of X. Then, if $x \in Y \cap Z$ we have $p(x) = x$ and $p(x) = 0$. Thus $Y \cap Z = \{0\}$. Further, notice that since $p(X) = Y$, it follows that $p^2 = p$. Hence, for any $x \in X$ we have that $p(x) \in Y$ and $x - p(x) \in Z$. Since $x = p(x) + (x - p(x))$ we have indeed shown that $X = Y \oplus Z$.

Conversely, assume that there exists a closed subspace Z of X such that $X = Y \oplus Z$. Define $p : X \to Y$ by $p(y + z) := y$ for any $y + z \in Y \oplus Z = X$. Then, clearly $p(X) = Y$ and $p(y) = y$ for any $y \in Y$. It remains to prove that p is in $\mathcal{B}(X, Y)$. It's clear that p is linear. That p is bounded is a consequence of the closed graph theorem. Indeed, if $(\ (x_n, y_n)\ )_{n=1}^{\infty}$ is a sequence, so that $p(x_n) = y_n$, converging to a point $(x, y) \in X \times Y$, then since $x_n - y_n \in Z$ for all n and Z is closed, it follows that $x - y \in Z$, whence $p(x) = y$ and we are done. ■

Lemma. *Let X a Banach Space and Y a subspace of X. If Y has finite dimension then Y is a complemented subspace of X.*

Proof. Since Y has finite dimension, we have that there is $n \in \mathbb{N}$, $y_1, \ldots, y_n \in Y$ such that Y is spanned by $y_1, \ldots, y_n$. For each $1 \leq k \leq n$, we define linear functionals $f_k : Y \to \mathbb{C}$ by

$$f_k \left(\sum_{j=1}^{n} \lambda_j y_j \right) := \lambda_k$$

Since all norms in finite dimension are equivalent, it's easy to check that each f_k is a bounded linear functional. Thus, as a consequence of the Hahn-Banach theorem, for each $1 \leq k \leq n$, there is a bounded linear functional $F_k : X \to \mathbb{C}$ such that $F_k|_Y = f_k$. Thus, if we define

$$p(x) := \sum_{j=1}^{n} F_j(x) y_j,$$

it follows that $p \in \mathcal{B}(X, Y)$, $p(X) = Y$ and that $p(y) = y$ for all $y \in Y$. So indeed Y is a complemented subspace. ■

Fredholm Operators

Recall that if $a : X \to Y$ is a linear map, the codimension of a in Y is given by $\mathrm{codim}(a(X)) := \dim(Y/a(X))$.

The following theorem is of major importance as it's saying that $(u - \lambda)$ is a Fredholm operator whenever $u \in \mathcal{K}(X)$ and $\lambda \in \mathbb{C} \setminus \{0\}$.

Theorem. *Let $u \in \mathcal{K}(X)$ and $\lambda \in \mathbb{C} \setminus \{0\}$. Then*

(1) $\ker(u - \lambda)$ *is finite dimensional.*

(2) $(u - \lambda)(X)$ *is closed and finite codimensional in X.*

Proof.

(1) Let $Y := \ker(u - \lambda)$. Notice that $Y = u(Y)$. Indeed, if $x \in Y$ then $u(x) = \lambda x$ and therefore $x = u(\lambda^{-1}x) \in u(Y)$. Conversely, if $y \in u(Y)$ there is $x \in Y$ such that $y = u(x) = \lambda x$. Hence, we get $u(y) = u(\lambda x) = \lambda u(x) = \lambda y$, so $y \in Y$. Then, since Y is itself a Banach space, we have that $u|_Y : Y \to Y$ is in $\mathcal{K}(Y)$. But $u|_Y = \lambda \mathrm{id}_Y$ and therefore id_Y is compact. But we already know that id_Y is compact if and only if Y is finite dimensional.

(2) Since $Y := \ker(u - \lambda)$ is finite dimensional by (1), we use a previous lemma to conclude that Y is a complemented subspace of X. Then, there is a closed subspace Z such that $X = Y \oplus Z$. by definition of Y we have that $(u-\lambda)(X) = (u-\lambda)(Z)$. Thus, to show that $(u-\lambda)(X)$ is closed, it suffices to prove that $(u-\lambda)|_Z$ is bounded below. Suppose on the contrary that it's not bounded below. Then, we know there must be a sequence of unit vectors $(z_n)_{n=1}^{\infty}$ in Z such that $(u-\lambda)(z_n) \to 0$ as $n \to \infty$. Since u is compact, there is a subsequence $(z_{n_k})_{k=1}^{\infty}$ such that $(u(z_{n_k}))_{k=1}^{\infty}$ is convergent, say to some $x \in X$. Then, since

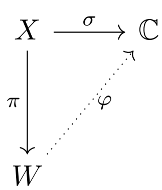
$$z_{n_k} = \frac{1}{\lambda}(u(z_{n_k}) - (u - \lambda)(z_{n_k}))$$

we have $z_{n_k} \to \frac{x}{\lambda}$. Thus, since Z is closed, we have that $x \in Z$. But, since u is continuous we also have $u(z_{n_k}) \to u(\frac{x}{\lambda})$. By uniqueness of limit we get $u(\frac{x}{\lambda}) = x$, which gives $u(x) = \lambda x$ and therefore that $x \in Y$. Therefore $x \in Y \cap Z = \{0\}$ and there fore $x = 0$, which is impossible because $\lambda z_{n_k} \to x$ and each z_{n_k} has norm 1. This contradiction shows that $(u-\lambda)|_Z$ is bounded below and therefore that $(u - \lambda)(X)$ is closed. We still need to prove that $(u - \lambda)(X)$ has finite codimension. We define $W := X/(u - \lambda)(X)$. Since $(u - \lambda)(X)$, W is a Banach space. Our aim is to show that $\dim(W) < \infty$

and we will do it by showing instead that $\dim(W^*) < \infty$. Let $\pi : X \to W$ be the canonical quotient map. It's clear that $\pi \in \mathcal{B}(X, W)$ and therefore we get $\pi^* \in \mathcal{B}(W^*, X^*)$. We claim that $\pi^*(W^*) = \ker(u^* - \lambda)$. Indeed, first we take any $\varphi \in W^*$ and compute $(u^* - \lambda)(\pi^*(\varphi))(x)$ for an arbitrary $x \in X$:

$$\begin{aligned}(u^* - \lambda)(\pi^*(\varphi))(x) &= (\pi \circ u)^*(\varphi)(x) - \lambda\pi^*(\varphi)(x) \\ &= \varphi(u(x) + [(u - \lambda)(X)]) - \lambda\varphi(x + [(u - \lambda)(X)]) \\ &= \varphi(u(x) - \lambda(x) + [(u - \lambda)(X)]) \\ &= \varphi(0 + [(u - \lambda)(X)]) \\ &= \varphi(0_W) = 0\end{aligned}$$

So $(u^* - \lambda)(\pi^*(\varphi)) = 0$ and therefore $\pi^*(W^*) \subseteq \ker(u^* - \lambda)$. For the reverse inclusion, we take any $\sigma \in \ker(u^* - \lambda)$. Notice that $\ker(\pi) \subseteq \ker(\sigma)$. Indeed, if $\pi(x) = 0$, then $x \in (u - \lambda)(X)$ and therefore $x = (u - \lambda)(x')$ for some $x' \in X$, whence $\sigma(x) = \sigma((u - \lambda)(x')) = (u^* - \lambda)(\sigma)(x) = 0$ because $\sigma \in \ker(u^* - \lambda)$. Thus, there is $\varphi \in W^*$ such that the following diagram commutes



That is, $\sigma = \varphi \circ \pi = \pi^*(\varphi) \in \pi^*(W^*)$. Hence $\pi^*(W^*) \supseteq \ker(u^* - \lambda)$ and this proves the claim. This gives $\dim(\pi^*(W^*)) = \dim(\ker(u^* - \lambda))$. But since $u^* \in \mathcal{K}(X^*)$, by (1) above we have that $\dim(\ker(u^* - \lambda)) < \infty$. Thus, $\dim(\pi^*(W^*)) < \infty$. Finally, we observe that π^* is injective, for if $\pi^*(\varphi) = \pi^*(\theta)$, then $\varphi([x]) = \theta([x])$ for any $[x] \in W$ and therefore $\varphi = \theta$. This gives

$$\dim(W^*) = \dim(\pi^*(W^*)) < \infty$$

as desired. ■

Definition. Let $u : X \to X$ be linear.

• One clearly has

$$\ker(u) \subseteq \ker(u^2) \subseteq \ker(u^3) \subseteq \dots$$

If $\ker(u^n) \neq \ker(u^{n+1})$ for all $n \in \mathbb{N}$, we say that u has **infinite ascent**; otherwise we say that u has **finite ascent** and we denote by ascent(u) to the least integer p such that $\ker(u^p) = \ker(u^{p+1})$.

• Similarly, one has

$$\operatorname{im}(u) \supseteq \operatorname{im}(u^2) \supseteq \operatorname{im}(u^3) \supseteq \dots$$

If $\operatorname{im}(u^n) \neq \operatorname{im}(u^{n+1})$ for all $n \in \mathbb{N}$, we say that u has **infinite descent**; otherwise we say that u has **finite descent** and we denote by descent(u) to the least integer p such that $\operatorname{im}(u^p) = \operatorname{im}(u^{p+1})$. ▲

Remark. One checks that $\ker(u^m) = \ker(u^n)$ for all $m, n \geq \text{ascent}(u)$ and that $\operatorname{im}(u^m) = \operatorname{im}(u^n)$ for all $m, n \geq \text{descent}(u)$. ▼

Next, we will show that $(u - \lambda)$ has both finite ascent and descent whenever $u \in \mathcal{K}(X)$ and $\lambda \in \mathbb{C} \setminus \{0\}$. But first, we need to recall the statement of Riesz's Lemma, a fundamental result from functional analysis.

Lemma. ***(Riesz)*** *If X is a normed vector space and $Y \subset X$ a closed proper subset, then for each $0 \leq \varepsilon \leq 1$ there is $x := x(\varepsilon)$ such that $\|x\| = 1$ and*

$$1 - \varepsilon \leq d(x, Y) := \inf_{y \in Y} \|x - y\| = \|x + Y\|_{X/Y}$$

Theorem. *Let $u \in \mathcal{K}(X)$ and $\lambda \in \mathbb{C} \setminus \{0\}$. Then $(u - \lambda)$ has finite ascent and descent.*

Proof. We first show that ascent$(u-\lambda) < \infty$. Suppose on the contrary that ascent$(u - \lambda) = \infty$. Let $N_n := \ker((u - \lambda)^n)$, then the inclusion $N_n \subset N_{n+1}$ is proper for each $n \in \mathbb{N}$. Since each N_n is closed, Riesz's Lemma gives the existence of a unit vector $x_n \in N_{n+1}$ so that

$$\|x_n + N_n\|_{N_{n+1}/N_n} \geq \frac{1}{2}$$

We get therefore a bounded sequence $(x_n)_{n=1}^{\infty}$ in X. If $m < n$, then

$$\begin{aligned} u(x_n) - u(x_m) &= u(x_n) - \lambda x_n - (u(x_m) - \lambda x_m) + \lambda x_n - \lambda x_m \\ &= \lambda x_n + [(u - \lambda)(x_n) - (u - \lambda)(x_m) - \lambda x_m] \\ &= \lambda x_n + z, \end{aligned}$$

where $z := [(u - \lambda)(x_n) - (u - \lambda)(x_m) - \lambda x_m]$. Since $x_n \in N_{n+1}$ and $x_m \in N_{m+1}$ we have $(u - \lambda)^k(x_n) = 0$ for $k \geq n + 1$ and $(u - \lambda)^l(x_m) = 0$ for any $l \geq m + 1$. Therefore,

$$(u - \lambda)^n z = (u - \lambda)^{n+1}(x_n) - (u - \lambda)^{n+1}(x_m) - \lambda(u - \lambda)^n(x_m) = 0$$

Thus, $z \in N_n$ and therefore

$$\|u(x_n) - u(x_m)\| = \|\lambda x_n + z\| \leq \|\lambda x_n + N_n\| = |\lambda| \|x_n + N_n\| \geq \frac{|\lambda|}{2}$$

This gives that $(u(x_n))_{n=1}^{\infty}$ is not Cauchy and therefore won't admit a convergent subsequence, contradicting that u is a compact operator. Hence, $(u - \lambda)$ has to have finite ascent.

Similarly, to show that descent$(u - \lambda) < \infty$ we assume for a contradiction that descent$(u-\lambda) = \infty$. Let $R_n := \mathrm{im}((u-\lambda)^n)$, then the inclusion $R_{n+1} \subset R_n$ is proper for every $n \in \mathbb{N}$. Now, we already know that $R_1 = (u-\lambda)(X)$ is closed, similarly $R_2 = (u-\lambda)^2(X)$ is closed because $(u-\lambda)^2 = u^2 - 2\lambda u + \lambda^2$, where $u^2 - 2\lambda u \in \mathcal{K}(X)$ and $-\lambda^2 \in \mathbb{C} \setminus \{0\}$. In general, each R_{n+1} is a closed proper subset of R_n. Thus, by Riesz's Lemma there is a unit vector $y_n \in R_n$ such that

$$\|y_n + R_{n+1}\|_{R_n/R_{n+1}} \geq \frac{1}{2}$$

This gives a bounded sequence $(y_n)_{n=1}^{\infty}$ in X. For $m < n$ we have now

$$u(y_m) - u(y_n) = \lambda y_m + s$$

where $s := [(u - \lambda)(y_m) - (u - \lambda)(y_n) - \lambda y_n]$. Now, since $R_n \subseteq R_{m+1} \subset R_m$ we have that $(u - \lambda)(y_m), (u - \lambda)(y_n) \in R_{m+1}$ and also $\lambda y_n \in R_{m+1}$. Hence $s \in R_{m+1}$ and therefore

$$\|u(y_m) - u(y_n)\| = \|\lambda y_m + s\| \leq \|\lambda y_m + R_{m+1}\| = |\lambda| \|y_m + R_{m+1}\| \geq \frac{|\lambda|}{2}$$

As before, this contradicts compactness of u, so we must have that $(u - \lambda)$ has finite descent. ■

Definition. An operator $u \in \mathcal{B}(X, Y)$ is said to be **Fredholm** if it has finite nullity and finite defect; where the nullity is defined by $\mathrm{nul}(u) := \dim(\ker(u))$ and the defect by $\mathrm{def}(u) := \mathrm{codim}(u(X))$. If u is Fredholm, its **Fredholm index** is defined by

$$\mathrm{ind}(u) := \mathrm{nul}(u) - \mathrm{def}(u)$$

We denote by $\mathcal{F}(X, Y)$ to all the Fredholm operators from X to Y. ▲

Lemma. *If $u \in \mathcal{F}(X, Y)$ then $u(X)$ is closed and there is a finite dimensional space $Z \subseteq Y$ such that $Y = u(X) \oplus Z$.*

Proof. Since u is Fredholm it follows that $Y/u(X)$ is finite dimensional. Then, there are $z_1, \dots, z_n \in Y$ such that $Y/u(X) = \mathrm{span}\{z_1 + u(X), \dots, z_n +$

$u(X)\}$. Set $Z := \operatorname{span}\{z_1, \ldots, z_n\} \subseteq Y$. Clearly $Z \cong Y/u(X)$. Suppose now that $y \in Z \cap u(X)$, that is $y = u(x) \in Z$ for some $x \in X$. Then, y gets mapped to 0, via the isomorphism from Z to $Y/u(X)$, so $y = 0$. That is, $Z \cap u(X) = \{0\}$, and since

$$\dim(Y) = \dim(Y/u(X)) + \dim(u(X)) = \dim(Z) + \dim(u(X)),$$

it follows that $Y = Z \oplus u(X)$. We still need to check that $u(X)$ is closed. Indeed, since Z is a finite dimensional subspace of Y, by a previous lemma, Z is a complemented subspace. That is, there is a closed subspace $W \subseteq Y$ such that $Y = Z \oplus W$. Thus, $Z \oplus W = Z \oplus u(X)$, so we get that $W = u(X)$, whence $u(X)$ is closed. ■

The following theorem is a fundamental result in Fredholm theory

Theorem. *Let $u \in \mathcal{F}(X, Y)$, $v \in \mathcal{F}(Y, Z)$. Then $vu \in \mathcal{F}(X, Z)$ and*

$$\operatorname{ind}(vu) = \operatorname{ind}(u) + \operatorname{ind}(v)$$

Proof. Set $Y_1 := \ker(v) \cap u(X)$. Since v is Fredholm, it follows that Y_1 is finite dimensional. Then, since $Y_1 \subseteq u(X)$, there is a closed subspace $Y_2 \subseteq u(X)$ so that

$$u(X) = Y_1 \oplus Y_2$$

Similarly, since $Y_1 \subseteq \ker(v)$ there is a closed subspace $Y_3 \subseteq \ker(v)$ such that

$$\ker(v) = Y_1 \oplus Y_3$$

Further, Y_3 has to be finite dimensional because $\ker(v)$ is. There's also a subspace $Y_4 \subseteq Y$ such that

$$Y = Y_1 \oplus Y_2 \oplus Y_3 \oplus Y_4 = u(X) \oplus Y_3 \oplus Y_4$$

and since u is Fredholm we must have that $Y_3 \oplus Y_4$ has finite dimension equal to $\operatorname{def}(u)$. Therefore, Y_4 is also finite dimensional. Next, we define a map $\varphi : \ker(vu) \to Y_1$ by

$$\varphi(x) := u(x)$$

By construction φ is surjective and $\ker(\varphi) = \ker(u)$. Hence, it follows that $\ker(vu)/\ker(u) \cong Y_1$ and therefore $\dim(Y_1) = \dim(\ker(vu)) - \dim(\ker(u))$. This gives

$$\operatorname{nul}(vu) = \dim(Y_1) + \operatorname{nul}(u) < \infty. \tag{1}$$

Now, since $\ker(v) = Y_1 \oplus Y_3$, we have $v(Y) = v(Y_2) \oplus v(Y_4)$. But also $v(Y_2) = v(Y_1 \oplus Y_2) = v(u(X)) = vu(X)$. Hence, $v(Y) = vu(X) \oplus v(Y_4)$.

Since v is Fredholm, there is a finite dimensional space $W \subseteq Z$ such that $Z = v(Y) \oplus W$ and $\dim(W) = \mathrm{def}(v)$. That is,

$$Z = vu(X) \oplus v(Y_4) \oplus W$$

Since Y_4 has finite dimension, $v(Y_4)$ is also finite dimensional. Thus,

$$\mathrm{def}(vu) = \dim(v(Y_4) \oplus W) = \dim(v(Y_4)) + \mathrm{def}(v) < \infty \tag{2}$$

Therefore, it follows from (1) and (2) that vu is indeed Fredholm. Finally, we define a map $\psi : Y_4 \to v(Y_4)$ by $\psi(y) := v(y)$. By definition ψ is surjective and since $\ker(v) \cap Y_4 = (Y_1 \oplus Y_3) \cap Y_4 = \{0\}$, it follows that ψ is also injective. Thus, $\dim(v(Y_4)) = \dim(V_4)$. So, again from (1) and (2) we have

$$\begin{aligned}
\mathrm{ind}(vu) &= \mathrm{nul}(vu) - \mathrm{def}(vu) \\
&= (\dim(Y_1) + \mathrm{nul}(u)) - (\dim(Y_4) + \mathrm{def}(v)) \\
&= \dim(Y_1) + \dim(Y_3) + \mathrm{nul}(u) - \dim(Y_4) - \dim(Y_3) - \mathrm{def}(v) \\
&= \dim(Y_1 \oplus Y_3) + \mathrm{nul}(u) - \dim(Y_4 \oplus Y_3) - \mathrm{def}(v) \\
&= \mathrm{nul}(v) + \mathrm{nul}(u) - \mathrm{def}(u) - \mathrm{def}(v) \\
&= \mathrm{ind}(u) + \mathrm{ind}(v)
\end{aligned}$$

■

The following theorem presents an immediate application of the Fredholm index.

Theorem. *Let $u \in \mathcal{K}(X)$ and $\lambda \in \mathbb{C} \setminus \{0\}$. Then*

(1) *$(u - \lambda)$ is Fredholm and $\mathrm{ind}(u - \lambda) = 0$.*

(2) *If $p = \mathrm{ascent}(u - \lambda)$, then*

$$X = \ker((u - \lambda)^p) \oplus (u - \lambda)^p(X)$$

Proof.

(1) We have already shown in a previous theorem that $\ker(u - \lambda)$ is finite dimensional and that $(u - \lambda)(X)$ is finite codimensional in X. Hence u is Fredholm. Recall also that both $\mathrm{ascent}(u - \lambda)$ and $\mathrm{descent}(u - \lambda)$ are finite. Take $m, n \in \mathbb{N}$ such that $m > n > \max\{\mathrm{ascent}(u - \lambda), \mathrm{descent}(u - \lambda)\}$. Then, $\mathrm{nul}((u - \lambda)^m) = \mathrm{nul}((u - \lambda)^n)$ and $\mathrm{def}((u - \lambda)^m) = \mathrm{def}((u - \lambda)^n)$. Hence, $\mathrm{ind}((u - \lambda)^m) = \mathrm{ind}((u - \lambda)^n)$, but by the previous theorem this

means that $m \cdot \operatorname{ind}(u-\lambda) = n \cdot \operatorname{ind}(u-\lambda)$. Since $m > n$, we must have $\operatorname{ind}(u-\lambda) = 0$, as wanted.

(2) Suppose that $x \in \ker((u-\lambda)^p) \cap (u-\lambda)^p(X)$. Then, $(u-\lambda)^p(x) = 0$ and $x = (u-\lambda)^p(y)$ for some $y \in X$. This gives $y \in \ker((u-\lambda)^{p+1})$ but since $p = \operatorname{ascent}(u-\lambda)$ we have $\ker((u-\lambda)^{p+1}) = \ker((u-\lambda)^p)$. Hence $x = (u-\lambda)^p(y) = 0$. This gives $\ker((u-\lambda)^p) \cap (u-\lambda)^p(X) = \{0\}$, so it suffices to show that $\dim(X) = \dim(\ker((u-\lambda)^p) \oplus (u-\lambda)^p(X))$. Indeed, it's known that

$$\dim(X) = \dim((u-\lambda)^p(X)) + \dim(X/(u-\lambda)^p(X)).$$

Now, by the previous theorem and (1) above, we have $\operatorname{ind}((u-\lambda)^p) = p \cdot \operatorname{ind}(u-\lambda) = 0$, so it follows that $\operatorname{nul}((u-\lambda)^p) = \operatorname{def}((u-\lambda)^p)$. Hence, $\dim(\ker(u-\lambda)^p) = \dim(X/(u-\lambda)^p(X))$, and therefore

$$\dim(X) = \dim((u-\lambda)^p(X)) + \dim(\ker(u-\lambda)^p)$$

as desired. ■

Remark. Notice that if $u : X \to Y$ is linear, then $\operatorname{nul}(u) = 0$ if and only if u is injective. Since $\operatorname{codim}(u(X)) = \dim(Y/u(X))$, $\operatorname{codim}(u(X)) = 0$ if and only if $u(X) = Y$. Hence, $\operatorname{def}(u) = 0$ if and only if u is surjective. This remark together with the previous theorem give at once the following result, known as the Fredholm Alternative. ▼

Corollary. *(Fredholm Alternative) If $u \in \mathcal{K}(X)$ and $\lambda \in \mathbb{C} \setminus \{0\}$, then $(u-\lambda)$ is injective if and only if it is surjective*

Proof. From the previous theorem we know that $\operatorname{ind}(u-\lambda) = 0$. Hence, $\operatorname{nul}(u-\lambda) = 0$ if and only if $\operatorname{def}(u-\lambda) = 0$. The result follows from our previous remark. ■

In the next theorem we will show, using Fredholm theory, a well known characterization of the spectrum of a compact operator. But first, we give a brief review of the spectrum and how it behaves under direct sums. Recall that if X is a Banach space and $u \in \mathcal{B}(X)$, the spectrum of u, denoted by $\sigma(u)$, is given by

$$\sigma(u) := \{\lambda \in \mathbb{C} : u - \lambda \text{ is not invertible } \}.$$

Suppose now that $X = Y \oplus Z$, where Y, Z are closed subspaces of X. Let $u \in \mathcal{B}(Y)$ and $v \in \mathcal{B}(Z)$. We get a map $u \oplus vX \to X$ by letting

$$(u \oplus v)(y+z) = u(y) + v(z)$$

Then $u \oplus v \in \mathcal{B}(X)$ because $u \oplus v = u \circ p_Y + v \circ (1 - p_Y)$, where p_Y and $1 - p_Y$ are bounded as they are the projections onto Y and Z respectively. Further, $u \oplus v$ is invertible if and only if u and v are invertible and $(u \oplus v)^{-1} = u^{-1} \oplus v^{-1}$. Hence,

$$\begin{aligned} \lambda \in \sigma(u \oplus v) &\iff (u \oplus v) - \lambda \ \text{ is not invertible} \\ &\iff (u - \lambda) \oplus (v - \lambda) \ \text{ is not invertible} \\ &\iff (u - \lambda) \ \text{ is not invertible} \quad \text{or} \quad (v - \lambda) \ \text{ is not invertible} \\ &\iff \lambda \in \sigma(u) \cup \sigma(v). \end{aligned}$$

So, it follows that $\sigma(u \oplus v) = \sigma(u) \cup \sigma(v)$.

Theorem. *If $u \in \mathcal{K}(X)$, then $\sigma(u)$ is countable, each non-zero point of $\sigma(u)$ is an eigenvalue of u and an isolated point of $\sigma(u)$.*

Proof. Fix $\lambda \in \sigma(u) \setminus \{0\}$. Then, $u - \lambda$ is not invertible, so the Fredholm alternative implies that $u - \lambda$ is not injective. That is, there is a non zero $x \in \ker(u - \lambda)$, whence $u(x) = \lambda x$. This gives of course that λ is an eigenvalue of u.

Now, let $p := \operatorname{ascent}(u - \lambda)$. We already proved that $p < \infty$ and that $X = Y \oplus Z$, where $Y := \ker((u - \lambda)^p)$ and $Z := (u - \lambda)^p(X)$ are both closed subspaces of X. We claim that $u|_Y \in \mathcal{B}(Y)$ and that $u|_Z \in \mathcal{B}(Z)$. To prove the claim, it suffices to show that $u(Y) \subseteq Y$ and that $u(Z) \subseteq Z$. Indeed, if $y \in Y$, then

$$\begin{aligned} (u - \lambda)^p(u(y)) &= (u - \lambda)^p(u(y) - \lambda y + \lambda y) \\ &= (u - \lambda)^{p+1}(y) + \lambda (u - \lambda)^p(y) \\ &= 0 \end{aligned}$$

because $y \in Y = \ker((u - \lambda)^p) = \ker((u - \lambda)^{p+1})$. Hence, $u(Y) \subseteq Y$. Similarly, if $z \in Z$, then $z = (u - \lambda)^p(x)$ for some $x \in X$; so

$$\begin{aligned} u(z) &= u\left((u - \lambda)^p(x)\right) \\ &= u\left((u - \lambda)^p(x)\right) - \lambda\left((u - \lambda)^p(x)\right) + \lambda\left((u - \lambda)^p(x)\right) \\ &= (u - \lambda)^{p+1}(x) + \lambda\left((u - \lambda)^p(x)\right), \end{aligned}$$

but since we always have $(u - \lambda)^{p+1}(X) \subset (u - \lambda)^p(X)$, the above equations gives in fact that $u(z) \in (u - \lambda)^p(X) = Z$. So indeed $u(Z) \subseteq Z$, proving the claim. Therefore,

$$u - \lambda = (u|_Y - \lambda \mathrm{id}_Y) \oplus (u_Z - \lambda \mathrm{id}_Z),$$

from where we get that

$$(u-\lambda)^p = (u|_Y - \lambda \mathrm{id}_Y)^p \oplus (u_Z - \lambda \mathrm{id}_Z)^p$$

But, by definition of Y, it follows that $(u|_Y - \lambda\mathrm{id}_Y)^p = 0$. Hence, there is a non zero $y \in Y$ such that $(u|_Y - \lambda\mathrm{id}_Y)(y) = 0$ and therefore $(u|_Y - \lambda\mathrm{id}_Y)$ is not invertible. That is, $\lambda \in \sigma(u|_Y)$, and therefore $\{\lambda\} \subset \sigma(u|_Y)$. On the other hand, if $\mu \in \sigma(u|_Y)$, there is a non zero $y \in Y$ such that $u|_Y(y) = \mu y$, whence $(u|_Y - \lambda\mathrm{id}_Y)(y) = (\mu - \lambda)y$, so $0 = (u|_Y - \lambda\mathrm{id}_Y)^p(y) = (\mu-\lambda)y$, which implies $\mu = \lambda$. This gives $\sigma(u|_Y) = \{\lambda\}$. We claim that $\lambda \notin \sigma(u|_Z)$. Well, since u is compact, it follows that $u|_Z \in \mathcal{K}(Z)$. Therefore, $(u_Z - \lambda\mathrm{id}_Z)^p$ is Fredholm. Notice that by definition of Y and Z, it follows that $\ker((u_Z - \lambda\mathrm{id}_Z)^p) = \{0\}$, so by the Fredholm alternative $(u_Z - \lambda\mathrm{id}_Z)^p$ must be invertible, whence $\lambda \notin \sigma(u|_Z)$ as we claimed. Thus,

$$\sigma(u) = \sigma(u|_Y \oplus u|_Z) = \{\lambda\} \sqcup \sigma(u|_Z).$$

This gives that $\sigma(u) \setminus \{\lambda\} = \sigma(u|_Z)$ is closed in $\sigma(u)$ and therefore λ is an isolated point of $\sigma(u)$.

Since $\lambda \in \sigma(u) \setminus \{0\}$ was arbitrary, it follows that that $\sigma(u) \setminus \{0\}$ consists of isolated points and hence it's a countable set. ■

Corollary. *If $\{\lambda_n : n \in \mathbb{N}\}$ is the non-zero spectrum of a compact operator, then*

$$\lim_{n\to\infty} \lambda_n = 0$$

Proof. If $\lambda := \lim_{n\to\infty} \lambda_n \neq 0$, then λ will be a non-isolated point of the spectrum, contradicting the previous theorem. ■

The following example is the original formulation of the Fredholm alternative, given by Erik Ivar Fredholm around 1900 while studying integral equations.

Example. Let $k \in C([0,1]^2)$, we have already seen that if $u \in \mathcal{B}(C[0,1])$, is given by

$$u(f)(s) := \int_0^1 k(t,s) f(t) dt,$$

then u is compact. Let $\lambda \in \mathbb{C} \setminus \{0\}$ and $g \in C([0,1])$. Consider the following two equations

$$(u-\lambda)(f) = g \tag{3}$$

$$(u-\lambda)(f) = 0 \tag{4}$$

Then, the Fredholm alternative says that either equation (3) has a unique solution or that equation (4) has non trivial solutions. Indeed, let $\{\lambda_n : 1 \leq n < N\}$ be the non zero spectrum of u, where $N \in \mathbb{N} \cup \{\infty\}$. Suppose first that $\lambda \neq \lambda_n$ for all n. Then, $(u - \lambda)$ is invertible and therefore equation (3) has a unique solution given by $f := (u - \lambda)^{-1}(g)$. However, if $\lambda = \lambda_n$ for some n, then $(u - \lambda)$ is not invertible, so the Fredholm alternative implies that $(u - \lambda)$ is not injective. That is, there is a non zero f such that $(u - \lambda)(f) = 0$, so we have a non trivial solution of (4). Moreover, since $(u - \lambda)$ is Fredholm, we know that $\ker((u - \lambda))$ is finite dimensional and therefore the solution set of equation (4) is finite dimensional. ▼

Definition. If $u : X \to Y$ is linear, a linear map $v : Y \to X$ is a **pseudoinverse** of u if $uvu = u$ ▲

Remark. If v is a pseudo inverse of u, then uv and vu are idempotents:

$$(uv)^2 = uvuv = uv \quad \text{and} \quad (vu)^2 = vuvu = vu$$

Further, we immediately check that $\ker(vu) = \ker(u)$ and $uv(Y) = u(X)$. ▼

We will show that any Fredholm operator has a pseudoinverse, but first we need a lemma.

Lemma. *Let $u : X \to Y$ be a linear map between normed spaces. If X is finite dimensional, then u is bounded.*

Proof. Define a norm in X by

$$\|x\|_0 = \max\{\|x\|, \|u(x)\|\}$$

Then, $\|u(x)\| \leq \|x\|_0$. Since X is finite dimensional, all norms are equivalent. Hence, there is a constant C such that $\|x\|_0 \leq C\|x\|$. Thus $\|u(x)\| \leq C\|x\|$, so u is bounded. ■

Theorem. *If $u \in \mathcal{F}(X, Y)$, then u admits a pseudoinverse v that is also Fredholm and such that $1 - uv$, $1 - vu$ have finite rank. Moreover, when $\operatorname{ind}(u) = 0$, v can be chosen to be invertible.*

Proof. Since u is Fredholm, there is a closed subspace $X_1 \subseteq X$ and a finite dimensional subspace $Y_1 \subseteq Y$ such that

$$X = \ker(u) \oplus X_1 \quad \text{and} \quad Y = u(X) \oplus Y_1$$

The map $u_1 := u|_{X_1}$ has trivial kernel and therefore it's an isomorphism from X_1 to $u(X)$. Further, u_1 is clearly bounded, so the open mapping theorem gives that the inverse $u_1^{-1} : u(X) \to X_1$ is also bounded. Let $v_1 := u_1^{-1}$. Suppose that $\text{ind}(u) = 0$, then $\text{nul}(u) = \text{def}(u)$ and therefore $\dim(\ker(u)) = \dim(Y/u(X)) = \dim(Y_1) < \infty$. Then, there is a linear isomorphism $w : Y_1 \to \ker(u)$. By the previous lemma, w is bounded. We define a map $v : Y \to X$ as follows: on $u(X)$ we set $v := v_1$, on Y_1 we set

$$v := \begin{cases} 0 & \text{if } \text{ind}(u) \neq 0 \\ w & \text{if } \text{ind}(u) = 0 \end{cases}$$

That is $v = v_1 \oplus 0$ if $\text{ind}(u) \neq 0$ and $v = v_1 \oplus w$ if $\text{ind}(u) = 0$. In any case v is a bounded linear map. Moreover, for any $x \in X$,

$$uvu(x) = u(v_1(u(x))) = u_1(v_1(u(x))) = u(x)$$

Hence, v is a pseudoinverse of u, which is invertible when $\text{ind}(u) = 0$ (because here $v = v_1 \oplus w$ and both v_1 and w are invertible). Notice that $\ker(v) \subseteq Y_1$ and therefore, since Y_1 is finite dimensional, $\text{nul}(v) < \infty$. Similarly, notice that $X_1 \subset v(Y)$, hence $\dim(X/v(Y)) < \dim(X/X_1) = \dim(\ker(u)) < \infty$, and therefore $\text{def}(v) < \infty$. This gives that v is indeed Fredholm. Finally, recall that vu is an idempotent, so

$$(1 - vu)(X) = \ker(vu) = \ker(u)$$

so $(1-vu)$ has finite rank. Similarly, notice that if $u(x)+y_1 \in u(X)\oplus Y_1 = Y$, then

$$(1-uv)(u(x)+y_1) = u(x)-y_1-uvu(x)-uv(y_1) = y_1-uv(y_1) = (1-uv)(y_1)$$

Hence, $(1 - uv)(Y) = (1 - uv)(Y_1)$ and since Y_1 is finite dimensional, it follows that $(1 - uv)$ also has finite rank. ■

Remark. Suppose that X is a finite dimensional Banach space. Then, all the elements of $\mathcal{B}(X)$ are Fredholm. Thus, the theory of Fredholm operators on finite dimensional spaces is not interesting. In what follows, we only care about infinite dimensional Banach spaces. ▼

Theorem. *(Atkinson) Let X be an infinite dimensional Banach space and $u \in \mathcal{B}(X)$. Then, $u \in \mathcal{F}(X)$ if and only if $u + \mathcal{K}(X)$ is invertible in the Calkin algebra $\mathcal{Q}(X) := \mathcal{B}(X)/\mathcal{K}(X)$.*

Proof. Suppose first that u is Fredholm. Let $\pi : \mathcal{B}(X) \to \mathcal{Q}(X)$ be the canonical quotient map. Then, by the previous theorem u has a pseudoinverse v such that $1 - uv$ and $1 - vu$ are finite rank operators and therefore elements of $\mathcal{K}(X)$. Then,

$$0 = \pi(1 - uv) = \pi(1) - \pi(uv) = 1_{\mathcal{Q}(X)} - \pi(u)\pi(v)$$

Hence, $\pi(u)\pi(v) = 1_{\mathcal{Q}(X)}$. Analogously we find $\pi(v)\pi(u) = 1_{\mathcal{Q}(X)}$, whence $\pi(u)^{-1} = \pi(v)$. That is, $\pi(u) = u + \mathcal{K}(X)$ is invertible in $\mathcal{Q}(X)$.

Conversely, assume that $\pi(u)$ is invertible in $\mathcal{Q}(X)$ with inverse given by $\pi(v)$ for some $v \in \mathcal{B}(X)$. Then, $uv = 1 + w_1$ and $vu = 1 + w_2$ for some $w_1, w_2 \in \mathcal{K}(X)$. Clearly $\ker(u) \subseteq \ker(1+w_2)$ and since w_2 is compact, $1+w_2$ is Fredholm and therefore $\mathrm{nul}(u) < \infty$. Similarly, w_1 is compact and hence $1 + w_1$ is Fredholm, so since $(1 + w_1)(X) \subseteq u(X)$ we have

$$\dim(X/u(X)) \leq \dim(X/(1 + w_1)(X)) < \infty,$$

which gives $\mathrm{def}(u) < \infty$. Therefore, u is Fredholm. ■

Theorem. *Let X be an infinite dimensional Banach space. Then, $\mathcal{F}(X)$ is open in $\mathcal{B}(X)$ and the index function* $\mathrm{ind} : \mathcal{F}(X) \to \mathbb{Z}$ *is continuous.*

Proof. By Atkinson's theorem, we have

$$\mathcal{F}(X) = \pi^{-1}\left(\mathrm{Inv}(\mathcal{Q}(X))\right)$$

It's well known that $\mathrm{Inv}(\mathcal{Q}(X))$ is open in $\mathcal{Q}(X)$ and since π is continuous, it follows that $\mathcal{F}(X)$ is open in $\mathcal{B}(X)$.

We prove that ind is continuous by showing that is locally constant. Let $u \in \mathcal{F}(X)$ and let $v \in \mathcal{F}(X)$ be a pseudoinverse for u. Then, $uvu = u$ and $1 - uv$, $1 - vu \in \mathcal{K}(X)$. Take $w \in \mathcal{F}(X)$ such that $w \in B_{\|v\|^{-1}}(u)$. Then,

$$\|uv - wv\| \leq \|u - w\|\|v\| \leq 1$$

Hence, $s := 1 - (uv - wv)$ is invertible in $\mathcal{F}(X)$ and therefore $\mathrm{ind}(s) = 0$. Furthermore,

$$su + u = su + uvu = u - (uvu - wvu) + uvu = wvu + u$$

Thus, $su = wvu$ and therefore $\mathrm{ind}(su) = \mathrm{ind}(wvu)$, which becomes

$$\mathrm{ind}(s) + \mathrm{ind}(u) = \mathrm{ind}(w) + \mathrm{ind}(v) + \mathrm{ind}(u),$$

whence $\mathrm{ind}(w) = -\mathrm{ind}(v)$. Since $w \in B_{\|v\|^{-1}}(u)$ was arbitrary, this gives $\mathrm{ind}|_{B_{\|v\|^{-1}}(u)} = -\mathrm{ind}(v)$, so ind is locally constant, as wanted. ■

Corollary. *Let X be an infinite dimensional Banach space, $u \in \mathcal{F}(X)$ and $w \in \mathcal{K}(X)$. Then,*

$$\operatorname{ind}(u + w) = \operatorname{ind}(u)$$

Proof. Atkinson's theorem implies that $u + tw \in \mathcal{F}(X)$ for any $t \in [0, 1]$, so we can define a function $\alpha : [0, 1] \to \mathbb{Z}$ by $\alpha(t) := \operatorname{ind}(u + tw)$. By the previous theorem, α is continuous. Since $[0, 1]$ is connected, $\alpha([0, 1])$ is a connected subset of $\mathbb{Z}$, and therefore $\alpha([0, 1])$ consists of a single point. This means that α is a constant function, so $\alpha(1) = \alpha(0)$, as we needed to prove. ■

Proposition. *Let $u \in \mathcal{F}(X)$. Then,* $\operatorname{ind}(u) = 0$ *if and only if u is the sum of an invertible operator and a compact one.*

Proof. Suppose first that $u = s + w$ where $s \in \operatorname{Inv}(\mathcal{B}(X))$ and $w \in \mathcal{K}(X)$. Since s is invertible, it has trivial kernel (whence $\operatorname{nul}(s) = 0$) and its range is X (that is $\operatorname{def}(s) = 0$). Thus, s is Fredholm and $\operatorname{ind}(s) = 0$. By the previous corollary we have

$$\operatorname{ind}(u) = \operatorname{ind}(s + w) = \operatorname{ind}(s) = 0$$

Now, assume that $\operatorname{ind}(u) = 0$. Then, there is an invertible pseudoinverse v of u. Hence, $\pi(u) = \pi(v^{-1})$, which means that $u = v^{-1} + w$ for some $w \in \mathcal{K}(X)$. ■

Remark. Is easy to find Fredholm operators of index 0 that are not invertible. For instance, if p is a finite rank non-zero idempotent, then $1 - p$ is Fredholm, of index 0 and not invertible. ▼

Recall that two operators $u, v \in \mathcal{B}(X)$ are said to be similar when there is an invertible operator s such that $u = svs^{-1}$. Two similar Fredholm operators must have the same index, whence the index is an obstruction to determine whether to operators are similar.

Example. Let $\mathcal{H}$ be a Hilbert space and $(e_n)_{n\in\mathbb{N}}$, $(f_n)_{n\in Z}$ be orthonormal bases for $\mathcal{H}$. The unilateral shift $u \in \mathcal{B}(\mathcal{H})$ is given by $u(e_n) = e_{n+1}$ for $n \in \mathbb{N}$, while the bilateral shift $v \in \mathcal{B}(\mathcal{H})$ is $u(f_n) = f_{n+1}$ for $n \in \mathbb{Z}$. Clearly $\operatorname{nul}(u) = 0$ but $\operatorname{def}(u) = \dim(\mathcal{H}/\operatorname{span}(e_n)_{n\geq 2}) = 1$. Therefore, $\operatorname{ind}(u) = -1$. Since v is invertible, $\operatorname{ind}(v) = 0$. This gives that u and v are not similar. Further, it's **not** possible to find an invertible s and a compact w such that $u = s + w$, as this will give $-1 = \operatorname{ind}(u) = \operatorname{ind}(s + w) = \operatorname{ind}(s) = 0$. ▼

Exercises

(Ex.1) Let X be a Banach space. If $p \in \mathcal{B}(X)$ is a compact idempotent, show that its rank is finite.

Proof. Since p is idempotent, we have $p(X) = \ker(1-p)$, but since p is compact, it follows that $1-p$ is Fredholm and therefore has finite dimensional kernel. Thus,

$$\operatorname{rank}(p) = \dim(p(X)) = \dim(\ker(1-p)) < \infty$$

as wanted. ■

(Ex.2) Let $u : X \to Y$ be a compact operator between Banach spaces. Show that if the range of u is closed, then it is finite-dimensional. (*Hint: Show that the well-defined operator* $X/\ker(u) \to u(X)$, $x + \ker(u) \mapsto u(x)$ *is an invertible compact operator.*)

Proof. We follow the hint. Let $a : X/\ker(u) \to u(X)$ be given by $a(x + \ker(u)) := u(x)$. It's clear that a is a well defined linear operator between Banach spaces. We now show that $a \in \mathcal{K}(X/\ker(u), u(X))$. Let $(x_n + \ker(u))_{n=1}^{\infty}$ be a bounded sequence, say by $M > 0$, in $X/\ker(u)$. For a fixed $n \in \mathbb{N}$, by definition of $\|x_n + \ker(n)\|$ there is a $y_n \in \ker(u)$ such that

$$\|x_n - y_n\| \leq M + 1$$

Therefore, $(x_n - y_n)_{n=1}^{\infty}$ is a bounded sequence in X, and since u is compact, the sequence $(u(x_n - y_n))_{n=1}^{\infty}$ admits a convergent subsequence in $u(X)$. But, since $y_n \in \ker(u)$ and by definition of a we have

$$(u(x_n - y_n))_{n=1}^{\infty} = (u(x_n))_{n=1}^{\infty} = (a(x_n + \ker(u)))_{n=1}^{\infty}$$

That is, a is compact. Further, notice that a has trivial kernel and that $a(X/\ker(u)) = u(X)$, whence a is invertible and therefore the open mapping theorem gives that a^{-1} is also bounded. This gives that $\mathrm{id}_{u(X)} = aa^{-1}$ is a compact operator in $\mathcal{B}(u(X))$, which implies that $u(X)$ is finite dimensional. ■

(Ex.3) Let X be an infinite dimensional Banach space. Show that if $u \in \mathcal{K}(X)$, then u is not Fredholm.

Proof. Recall that $u \in \mathcal{F}(X)$ if and only if $u + \mathcal{K}(X)$ is invertible in $\mathcal{Q}(X)$. Since $u \in \mathcal{K}(X)$, it follows that $u + \mathcal{K}(X) = \mathcal{K}(X)$, which is not invertible in $\mathcal{Q}(X)$. ■

(Ex.4) Let X, Y be Banach spaces and suppose that $u \in \mathcal{B}(X,Y)$ has compact transpose u^*. Show that u is compact using the fact that u^{**} is compact.

Proof. Since $u^* \in \mathcal{K}(Y^*, X^*)$, it follows that $u^{**} \in \mathcal{K}(X^{**}, Y^{**})$. Now, recall that there is an isometric embedding map $\widehat{\cdot}: X \hookrightarrow X^{**}$ given by

$$\widehat{x}(\varphi) := \varphi(x) \quad \forall x \in X \ \forall \varphi \in X^*$$

Now, let $(x_n)_{n=1}^{\infty}$ be a bounded sequence in X. Then, $(\widehat{x_n})_{n=1}^{\infty}$ is a bounded sequence in X^{**}. Since u^{**} is compact, it follows that $(u^{**}(\widehat{x_n}))_{n=1}^{\infty}$ admits a convergent subsequence in Y^{**}. Now notice that for any $\sigma \in Y^*$, we have

$$u^{**}(\widehat{x})(\sigma) = \widehat{x}(u^*(\sigma)) = \widehat{x}(\sigma \circ u) = \sigma(u(x)) = \widehat{u(x)}(\sigma)$$

That is, $u^{**}(\widehat{x}) = \widehat{u(x)}$ and therefore $\|u^{**}(\widehat{x_n})\| = \|u(x_n)\|$, which implies that $(u(x_n))_{n=1}^{\infty}$ admits a convergent subsequence, so u is in fact compact. ■

(Ex.5) Let $u : X \to X'$ and $u' : X' \to Y'$ be bounded operators between Banach spaces. Show that the linear map

$$u \oplus u' : X \oplus X' \to Y \oplus Y', \quad (x, x') \mapsto (u(x), u'(x')),$$

is bounded with norm $\max\{\|u\|, \|u'\|\}$. Show that if u and u' are Fredholm operators, so is $u \oplus u'$, and $\operatorname{ind}(u \oplus u') = \operatorname{ind}(u) + \operatorname{ind}(u')$.

Proof. We equiv the direct sum of Banach spaces with the max norm. Then,

$$\begin{aligned}
\|u \oplus u'\| &= \sup_{\|(x,x')\| \le 1} \|(u(x), u'(x'))\| \\
&= \sup_{\|(x,x')\| \le 1} \max\{\|u(x)\|, \|u'(x')\|\} \\
&= \max\{\sup_{\|x\| \le 1} \|u(x)\|, \sup_{\|x'\| \le 1} \|u'(x)\|\} \\
&= \max\{\|u\|, \|u'\|\}
\end{aligned}$$

Now, let $N := \ker(u)$, $N' := \ker(u')$, $C := Y/u(X)$ and $C' := Y'/u'(X')$. Since both u and u' are Fredholm, it follows that N, N', C and C' are all finite dimensional spaces. Notice that $\ker(u \oplus u') = N \oplus N'$ and therefore $\operatorname{nul}(u \oplus u') = \operatorname{nul}(u) + \operatorname{nul}(u') < \infty$. Similarly,

$$(Y \oplus Y')/((u \oplus u')(X \oplus X')) \cong C \oplus C'$$

Hence, $\operatorname{def}(u \oplus u') = \operatorname{def}(u) + \operatorname{def}(u') < \infty$. That is, $u \oplus u$ is Fredholm and

$$\operatorname{ind}(u \oplus u') = \operatorname{ind}(u) + \operatorname{ind}(u')$$

as wanted. ■

(Ex.6) If X is an Infinite-dimensional Banach space and $u \in \mathcal{B}(X)$, show that

$$\bigcap_{v \in \mathcal{K}(X)} \sigma(u+v) = \sigma(u) \setminus \{\lambda \in \mathbb{C} : u - \lambda \in \mathcal{F}(X),\ \operatorname{ind}(u-\lambda) = 0\}$$

Proof. We prove this by double inclusion.

Suppose first that $\lambda \in$LHS. That is, $\lambda \in \sigma(u+v)$ for all $v \in \mathcal{K}(X)$. In particular, $0 \in \mathcal{K}(X)$ so we have $\lambda \in \sigma(u)$. Assume, for the sake of contradiction, that $u - \lambda \in \mathcal{F}(X)$ with $\operatorname{ind}(u-\lambda) = 0$. Then, there is a pseudo inverse $w \in \operatorname{inv}(\mathcal{B}(X))$ such that $1-(u-\lambda)w$ is compact. Thus, $v := w^{-1}-(u-\lambda)$ is a compact operator, and therefore $u+v-\lambda = w^{-1}$ is invertible, contradicting that $\lambda \in \sigma(u+v)$. Hence, $\lambda \in$RHS.

Assume now that $\lambda \in$RHS. That is, $\lambda \in \sigma(u)$ but $\lambda \notin \{\mu \in \mathbb{C} : u - \mu \in \mathcal{F}(X),\ \operatorname{ind}(u-\mu) = 0\}$. Assume on the contrary that $\lambda \notin$LHS. Then, there is $v \in \mathcal{K}(X)$ such that $\lambda \notin \sigma(u+v)$. That is, $u+v-\lambda$ is invertible. This gives of course that $\operatorname{ind}(u+v-\lambda) = 0$. Suppose that $(u-\lambda)$ is Fredholm, then since v is compact

$$\operatorname{ind}(u-\lambda) = \operatorname{ind}(u+v-\lambda) = 0,$$

which is impossible because $\lambda \notin \{\mu \in \mathbb{C} : u - \mu \in \mathcal{F}(X),\ \operatorname{ind}(u-\mu) = 0\}$. Suppose now that $(u-\lambda)$ is not Fredholm, that is $\pi(u-\lambda)$ is not invertible. But, $\pi(u-\lambda) = \pi(u+v-\lambda)$ which is invertible because $\lambda \notin \sigma(u+v)$, so this is also impossible. We've reached a contradiction and therefore $\lambda \in$LHS. ■

(Ex.7) Let X be an infinite dimensional Banach space and $u \in \mathcal{B}(X)$. Define the **essential spectrum**, $\sigma_e(u)$, by

$$\sigma_e(u) := \{\lambda \in \mathbb{C} : (u-\lambda) \notin \mathcal{F}(X)\}$$

Show that $\sigma_e(u)$ is a non-empty compact subset of $\sigma(u)$

Proof. Notice that

$$\begin{aligned}\sigma_e(u) &= \{\lambda \in \mathbb{C} : \pi(u-\lambda) \text{ is not invertible in } \mathcal{Q}(X)\} \\ &= \{\lambda \in \mathbb{C} : \pi(u) - \lambda \text{ is not invertible in } \mathcal{Q}(X)\} \\ &= \sigma_{\mathcal{Q}(X)}(\pi(u))\end{aligned}$$

Thus, $\sigma_e(u)$ is compact and non-empty. Since $\operatorname{Inv}(\mathcal{B}(X)) \subset \mathcal{F}(X)$, it follows that if $\lambda \in \sigma_e(u)$, then $\lambda \in \sigma(u)$, so indeed $\sigma_e(u) \subseteq \sigma(u)$. ■

(Ex.8) Suppose that $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces. Show that $u \in \mathcal{F}(\mathcal{H}_1, \mathcal{H}_2)$ if and only if $u(\mathcal{H}_1)$ is closed in $\mathcal{H}_2$, and $\ker(u)$, $\ker(u^*)$ are both finite dimensional spaces. Conclude that $\operatorname{ind}(u) = \operatorname{nul}(u) - \operatorname{nul}(u^*)$.

Proof. Notice that it suffices to show that $\ker(u^*)$ is isomorphic to $\mathcal{H}_2/u(\mathcal{H}_1)$ whenever $u(\mathcal{H}_1)$ is closed. We define a map $\Phi : \ker(u^*) \to \mathcal{H}_2/u(\mathcal{H}_1)$ by $\Phi(\eta) := \eta + u(\mathcal{H}_1)$. We prove Φ is an isomorphism. It's is clearly linear. If $\eta \in \ker(u^*)$ is such that $\Phi(\eta) = u(\mathcal{H}_1)$, we have $\eta = u(\xi)$ for some $\xi \in \mathcal{H}_1$ and therefore

$$\|\eta\|^2 = \langle \eta, \eta \rangle = \langle u(\xi), \eta \rangle = \langle \xi, u^*(\eta) \rangle = \langle \xi, 0 \rangle = 0$$

Hence, $\eta = 0$. This gives that Φ is injective. To prove surjectivity, we claim first that $\ker(u^*)^\perp = u(\mathcal{H}_1)$. Indeed, the inclusion $u(\mathcal{H}_1) \subseteq \ker(u^*)^\perp$ is easy. For the reverse one, notice that it's also easy to see that $u(\mathcal{H}_1)^\perp \subseteq \ker(u^*)$. Hence, since $u(\mathcal{H}_1)$ is closed it follows taking orthogonal complements that $\ker(u^*)^\perp \subseteq u(\mathcal{H}_1)$. This proves the claim. Now, since $\mathcal{H}_2 = \ker(u^*) \oplus \ker(u^*)^\perp$, we have in fact $\mathcal{H}_2 = \ker(u^*) \oplus u(\mathcal{H}_1)$. Take $\eta + u(\mathcal{H}_1) \in \mathcal{H}_2/u(\mathcal{H}_1)$ with $\eta = \eta_1 + \eta_2 \in \ker(u^*) \oplus u(\mathcal{H}_1)$, it follows that $\Phi(\eta_1) = \eta + u(\mathcal{H}_1)$, whence Φ is surjective. ■